On the structure of (pan, even hole)-free graphs

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Outline

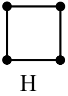
1. H-Free Graphs
2. Decomposition Theorem
3. Applications

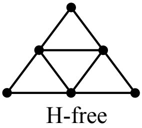
Outline

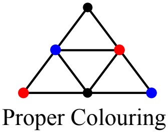
H-Free Graphs

Definition

For a graph H, a graph G is H-free, when G **does not contain** H as an **induced** subgraph.

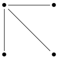




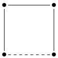


Claws, holes, and pans

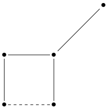
We will deal with (claw, even hole)-free graphs
Even holes: holes with even length
(Pan, even hole)-free graphs
S. Olariu introduces pan, proved SPGC for pan-free graphs.
Stability number of pan-free graphs is in P
(Brandstadt, Lozin, Mosca)



claw



hole



pan

Related Graph Classes and Recognition

Chordal: G is hole-free; i.e., $(C_4, C_5, \ldots)$-free: **linear time** [Rose, Tarjan, Lueker; SIAM JComp 1976]
Odd-hole-free; i.e., $(C_5, C_7, \ldots)$-free: **OPEN**
Perfect: (odd-hole,odd-anti-hole)-free: **polytime** [Chudnovsky, Cornuejols, Liu, Seymour, Vušković; Combinatorica 2005]
Even-hole-free; i.e., $(C_4, C_6, \ldots)$-free: **polytime** (more on this)

Note: Information System on Graph Classes `http://www.graphclasses.org/` defines even-hole free as $(C_6, C_8, \ldots)$-free. C_4 is not excluded, but we do exclude C_4!!

Finding Even-Holes

- $\mathcal{O}(n^{40})$ [Conforti, Cornuéjols, Kapoor, and Vušković; JGT 2002].
- $\mathcal{O}(n^{31})$ [Chudnovsky, Kawarabayashi, and Seymour; JGT 2005].
- $\mathcal{O}(n^{19})$ [da Silva and Vušković; JCTB 2013].
- $\mathcal{O}(m^3 n^5)$ [Chang and Lu; SODA 2012, arxiv 2013].

- In planar: $\mathcal{O}(n^3)$ [Porto; LATIN 1992]
- In claw-free: $\mathcal{O}(n^8)$ [van 't Hof, Kamiński, Paulusma; Algorithmica 2012]
- In circular-arc: $\mathcal{O}(mn^2 loglogn)$ [Cameron, Eschen, Hoàng, Sritharan; 2007]

Combinatorial Optimization Problems

	Clique	Ind. Set	Colouring	Clique cove
Even-Hole-free	P	?	?	NP-hard
Odd-Hole-free	NP-hard	P	NP-hard	??
Pan-free	NP-hard	P	NP-hard	NP-hard
(Pan, Even hole)-free	P	P	P	??

Even-hole-free graphs: $\chi(G) \leq 2\omega(G) - 1$ [Addario-Berry, Chudnovsky, Havet, Reed, Seymour; JCTB 2008]

Our Results

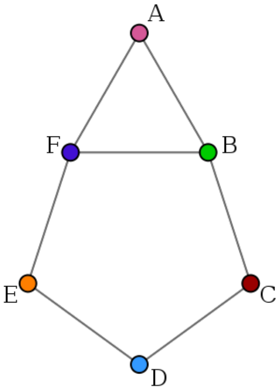
Theorem

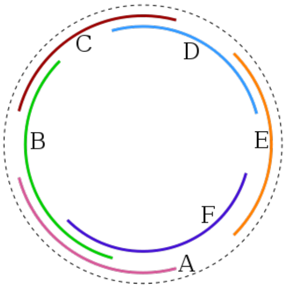
For a (pan,even-hole)-free graph G, one of the following hold:

1. *G is a clique.*
2. *G contains a clique cutset.*
3. *G is a unit circular arc graph*
4. *G is the join of a clique and a unit circular arc graph.*

- Recognition in $\mathcal{O}(nm)$ time.
- Colouring in $\mathcal{O}(n^{2.5} + nm)$ time.

Circular Arc Graphs





"unit" means all arcs have the same length

Colouring is NP-complete for circular arc graphs

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Our Decomposition Theorem

Theorem

For a (pan,even-hole)-free graph G, one of the following hold:

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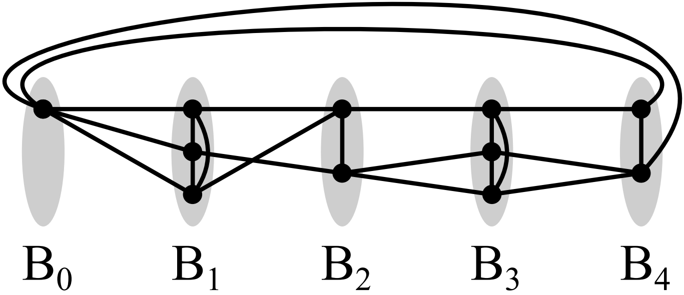
- Auxiliary structure generalizing holes: buoy

Holes and Buoys

A length ℓ-***buoy*** has ℓ bags: $B_0, \ldots, B_{\ell-1}$, each bag is a clique, and each vertex in a bag has neighbours in adjacent bags (but not other bags).



Our Decomposition Theorem

Theorem

For a (pan,even-hole)-free graph G, one of the following hold:

1. *G is a clique.*
2. *G contains a clique cutset.*
3. *G is a **buoy****
4. *G is the join of a clique and a **5-buoy****.

* These buoys are extremely special, as we will see.

Structure of a Buoy

Theorem

If B is a ℓ-buoy in a (pan,even-hole)-free graph, then:

- *Each B_i can be ordered by neighbourhood inclusion.*
- *Either $(B_i \cup B_{i+1})$ or $B_i \cup B_{i-1}$ is a clique.*

For efficient recognition:

Theorem

If B is an ℓ-buoy where each B_i can be ordered by neighbourhood inclusion, then every hole in B has length ℓ.

Structure of a Buoy

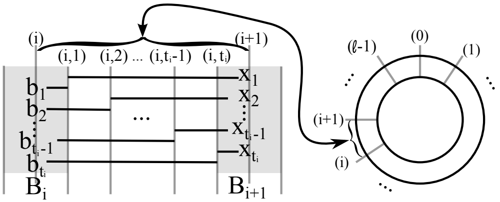
Theorem

If B is a ℓ-buoy in a (pan,even-hole)-free graph, then:

- *Each B_i can be ordered by neighbourhood inclusion.*
- *Either $(B_i \cup B_{i+1})$ or $(B_i \cup B_{i-1})$ is a clique.*

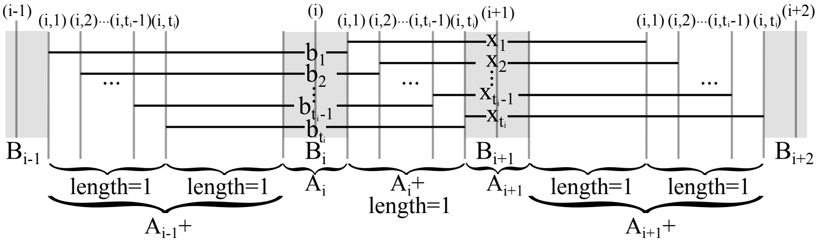
Buoys To Circular Arcs

Remember: each bag is orderable by neighbourhood inclusion.



Buoys To Unit Circular Arcs

Case: $B_i \cup B_{i+1}$ is not a clique.
Remember: every other pair of bags is a clique.



Each arc will have length 2+ϵ
Arc A_i has length ϵ

Neighbourhood of a buoy

Theorem

Let B be an ℓ-buoy in a (pan,even-hole)-free graph and let x be a neighbour B. Then:

- *x adjacent to 5 bags implies $\ell = 5$ and x universal to B (*).*
- *x adjacent to 2 bags implies these bags are consecutive and form a clique.*
- *x adjacent to 3 bags implies these bags are consecutive and x universal to the middle bag.*

(*) These vertices are the only way we have a unit circular arc graph joined with a clique in our decomposition.

Neighbourhood of a buoy

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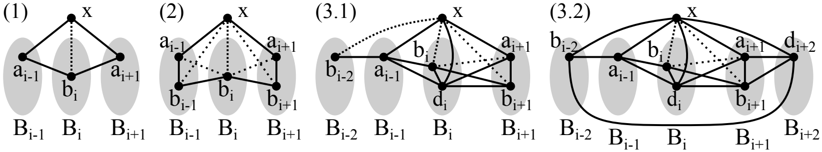


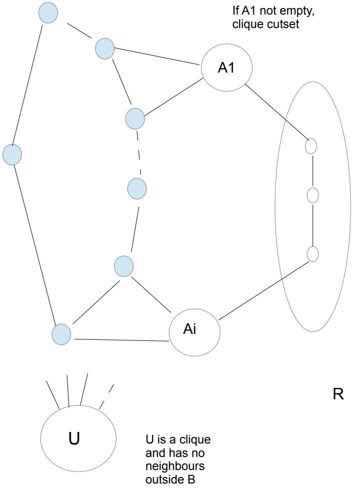
Decomposition Theorem

Theorem

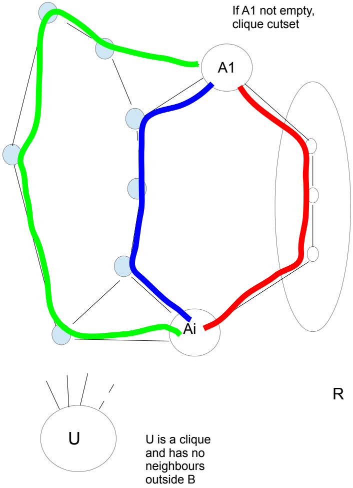
Consider a (pan,even-hole)-free graph G. Let B be a "maximal" buoy of G:

1. *B contains all vertices of G*
2. *G contains a clique cutset.*
3. *G is the join of a clique and a 5-buoy (unit circular arc graph).*

Structure of Maximal Buoys

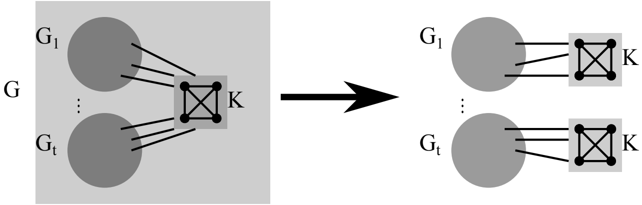


Structure of Maximal Buoys



Outline

Main Tool Clique Cutset Decomposition



- Computation in $\mathcal{O}(nm)$ time with $< n$ ***atoms*** [Tarjan; JDM 1985]
- Applications: Chromatic number, and the presence of a hole. [Whitesides; 1984]

Colouring

Note: only need to consider atoms and our atoms are unit circular arc graphs.

1. Run Clique Cutset decomposition : $\mathcal{O}(nm)$ time, with $< n$ atoms
2. Colour the atoms of the decomposition: $\mathcal{O}(n^{1.5} + m)$ per atom.
3. Now, $\chi(G) = \max\{\chi(H) : H \text{ is an atom of } G\}$.

Total time: $\mathcal{O}(n^{2.5} + nm)$.
χ-bounded: $\chi(G) \leq 1.5\,\omega(G)$.

Unit Circular Arc representation construction: $\mathcal{O}(n + m)$ [Lin, Szwarcfiter; SIAM JDM 2008]
Unit Circular Arc colouring from a representation: $\mathcal{O}(n^{1.5})$ [Shih, Hsu; JDAM 1989]

Recognition

1. Run Clique Cutset decomposition : $\mathcal{O}(nm)$ time, with $< n$ atoms
2. For each atom:
3. verify that no holes of the atom form a pan with a vertex outside
5. Build our special buoy B from this hole: $\mathcal{O}(n+m)$.
6. If B cannot be built, we produce a pan or an even hole
7. Build an unit circular arc representation.

Total time: $\mathcal{O}(nm)$.

Chordality Testing: $O(n+m)$: [Rose, Tarjan, Lueker; SIAM JComp 1976]

Concluding Remarks

(pan,even-hole)-free graphs decompose into *almost* unit-circular arc graphs by clique cutsets. This allows:

- Recognition in $\mathcal{O}(nm + m^{1.69})$ time.
- Colouring in $\mathcal{O}(n^{2.5} + nm)$ time.
- Bounding parameters : $\omega(G) \leq \chi(G) \leq 1.5\omega(G)$.

Open Problems:

- Odd-hole-free: recognition, independent set, structural characterization.
- Even-hole-free: independent set, colouring.
- (pan,even-hole)-free: clique cover.
- Characterize circular arc graphs by minimal forbidden induced subgraphs.